A multilevel preconditioner for the mortar method for nonconforming P1 finite element

Talal Rahman and Xuejun Xu

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Bergen Center for Computational Science, Høyt.teknologisenteret,
Thormøhlen gate 55, N-5008 Bergen, Norway
A multilevel preconditioner for the mortar method for nonconforming P1 finite element

Talal Rahman ∗ Xuejun Xu †

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Abstract

A multilevel preconditioner based on the abstract framework of the auxiliary space method, is developed for the mortar method for the nonconforming P1 finite element or the lowest order Crouzeix-Raviart finite element on nonmatching grids. It is shown that the proposed preconditioner is quasi-optimal, in the sense that the condition number of the preconditioned system is independent of the mesh size, and depends only quadratically on the number of refinement levels. Some numerical results confirming the theory are also provided.

1 Introduction

The mortar finite element method is a special domain decomposition methodology, which appears to be very attractive to the scientific computing community since it can handle situations where meshes on different subdomains need not align across the interfaces. The matching between the discretizations on adjacent subdomains is only enforced weakly. In [6], Bernardi, Maday and Patera first introduced the basic concept of mortar element methods, and applied it for coupling spectral elements with finite elements. Since then, the methodology has been extensively used and analyzed by many authors. In [4], Belgacem studied the mortar element method under a primal hybrid finite element formulation. Meanwhile, some extensions to the three dimensional problem, and to using dual basis for the Lagrange multiplier space were considered in [5, 7, 16, 24]. Recently, much work has been devoted towards constructing efficient iterative solvers for the discrete system resulting from the mortar element method. The first approaches were based on the iterative substructuring method, see for instance [1, 2, 3, 12]. Multigrid methods for the mortar element method have also been considered. Gopalakrishnan and Pasciak [14] presented a variable

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∗BCCS, Bergen Center for Computational Science, Thormøhlensgt. 55, 5008 Bergen, Norway (Email: t.alal@bccs.uib.no).
†LSEC, Institute of Computational Mathematics, Chinese Academy of Sciences, P.O.Box 2719, Beijing, 100080, People’s Republic of China (Email: xxj@lsec.cc.ac.cn).
V-cycle multigrid, while Braess, Dahmen and Wieners [8], and Wohlmuth [25] established a W-cycle multigrid based on the hybrid formulation which gives rise to a saddle point problem. We note that Braess, Deuflhard and Lipnikov [9] have recently constructed a subspace cascadic multigrid method for the mortar element method based on a saddle point formulation.

There have been some interests in the construction and implementation of the mortar element method for the lowest order Crouzeix-Raviart (CR) finite element or the nonconforming P1 finite element. Marcinkowski [17] first presented the mortar element method for the CR element. Based on this method, Xu and Chen [29] introduced an optimal W-cycle multigrid method for the discrete system, with a convergence rate which is independent of the mesh size and the level of refinements. Recently developed additive Schwarz methods for elliptic problems with discontinuous coefficients using the CR mortar finite element can be found in [18, 20].

The objective of this paper is to propose an effective multilevel preconditioner for the CR mortar finite element. Using the so-called auxiliary space technique developed in [26, 19], we propose a multilevel preconditioner for the CR mortar finite element method. We choose the conforming P1 mortar finite element space as the auxiliary space. A recently developed effective multilevel preconditioner for the conforming P1 mortar finite element, cf. [13], is used as the preconditioner for the auxiliary space. The new multilevel preconditioner is shown to have the same quasi-optimal convergence behavior as the auxiliary preconditioner. The condition number of the preconditioned system is independent of the mesh size, and only quadratically dependent on the number of refinement levels.

The rest of this paper is organized as follows. In Section 2, we introduce our discrete problems, in Section 3 we describe the multilevel preconditioner for the conforming P1 mortar finite element. In Section 4, we construct our multilevel preconditioner for the CR mortar finite element. Condition number estimate of the multilevel preconditioner will be given in Section 5. In the last section, some numerical results supporting the theory will be presented.

2 Mortar method for the nonconforming P1 finite element

For simplicity, we consider the following model problem

\[
\begin{cases}
-\Delta u = f & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\]  

(1)
where \( \Omega \subset \mathbb{R}^2 \) is a bounded polygonal domain, \( f \in L^2(\Omega) \). It is not difficult to extend the results of this paper to a more general second order elliptic problem. The variational formulation of the problem (1) is to find \( u \in H^1_0(\Omega) \) such that

\[
a(u, v) = (f, v) \quad \forall v \in H^1_0(\Omega),
\]

where the bilinear form \( a(\cdot, \cdot) \) is given as

\[
a(u, v) = \int_{\Omega} \nabla u \cdot \nabla v \, dx \quad \forall u, v \in H^1(\Omega)
\]

and \( (f, v) = \int_{\Omega} fv \, dx \).

We now introduce the mortar finite element method of [17] for solving (1), where the nonconforming P1 finite element is used for the discretization. Let \( \Omega \) be partitioned into a set of nonoverlapping polygonal subdomains \( \{\Omega_i\} \) such that \( \Omega = \bigcup_{i=1}^{N} \Omega_i \) and \( \Omega_i \cap \Omega_j = \emptyset, \quad i \neq j \). We consider only the case where the partition is geometrically conforming, that is, the subdomains are arranged so that the intersection \( \Omega_i \cap \Omega_j \) for \( i \neq j \) is either an empty set, an edge or a vertex. The skeleton \( \Gamma = \bigcup_{i=1}^{N} \partial \Omega_i \setminus \partial \Omega \) is decomposed into a set of disjoint open straight segments \( \gamma_m \) \((1 \leq m \leq M)\) (that are the edges of subdomains) called the mortars, such that

\[
\Gamma = \bigcup_{m=1}^{M} \bar{\gamma}_m, \quad \gamma_m \cap \gamma_n = \emptyset, \quad \text{if} \quad m \neq n.
\]

If \( \gamma_m \) is an open edge of a subdomain \( \Omega_i \), we refer to it as a mortar side of \( \Omega_i \), and denote it by \( \gamma_{m(i)} \). Consequently, the other side of \( \gamma_{m(i)} \), which is an edge of another subdomain say \( \Omega_j \), occupying the same geometrical space as that of \( \gamma_{m(i)} \), will be referred to as the corresponding nonmortar side of \( \Omega_j \), and will be denoted as \( \delta_{m(j)} \).

For \( i = 1, \cdots, N \), let \( T_{1,i} \) be the initial shape regular triangulation of \( \Omega_i \) with the mesh size \( h_1 \). The overall triangulation generally does not match at the subdomain interfaces. Let the global mesh \( \bigcup_{i=1}^{N} T_{1,i} \) be denoted by \( T_1 \). We refine the triangulation \( T_1 \) to produce \( T_2 \) by joining the edge midpoints of the triangles in \( T_1 \). The mesh size \( h_2 \) in the triangulation \( T_2 \) is then given by \( h_2 = h_1/2 \). Repeating the process for \( l \) times, \( l = 1, \cdots, N \), we get an \( l \) times refined triangulation \( T_l \) with the mesh size \( h_l = h_1 2^{-l} \) \((l = 1, \cdots, L)\).

At the finest level \( L \), locally in each subdomain \( \Omega_i \), we use the nonconforming P1 or the lowest order CR finite element space, \( V_{L,i} \), whose functions are piecewise linear on the triangulation \( T_{L,i} \), determined uniquely by their values at the edge midpoints, and vanishing at the edge midpoints of the boundary \( \partial \Omega \). The sets of edge midpoints, also referred to as
the nonconforming P1 or the CR nodal points, those belonging to \( \Gamma_i, \partial \Omega_i \) and \( \partial \Omega \) are denoted by \( \Omega_{L;i}^{CR}, \partial \Omega_{L;i}^{CR} \) and \( \partial \Omega_{L;i}^{CR} \), respectively. Consequently, the functions of \( V_{L,i} \) are piecewise linear on each triangle of \( T_{L,i} \), continuous at the CR nodes of \( \Omega_{L;i}^{CR} \), and equals to zero at the CR nodes of \( \partial \Omega_{L;i}^{CR} \cap \partial \Omega_L \).

Let

\[
\tilde{V}_L = \prod_{i=1}^{N} V_{L,i} = \{ v_L | v_{L|i} = v_{L,i} \in V_{L,i} \}.
\]

Due to nonmatching grids along subdomain interfaces, each interface \( \gamma_m \), where \( \gamma_m = \gamma_m^{(i)} = \delta_{m(j)}, 1 \leq m \leq M \), inherits two independent 1D triangulations \( T_L(\gamma_m^{(i)}) \) and \( T_L(\delta_{m(j)}) \). Consequently, there are two sets of CR nodes belonging to \( \gamma_m \), the midpoints of the elements belonging to \( T_L(\gamma_m^{(i)}) \) and \( T_L(\delta_{m(j)}) \), we denote them by \( \gamma_{L,m}^{(i)} \) and \( \delta_{L,m(j)}^{CR} \), respectively. Additionally, we define an auxiliary test space as \( S_L(\delta_{m(j)}) \) defined by

\[
S_L(\delta_{m(j)}) := \{ v | v \in L^2(\delta_{m(j)}), \text{ and } v \text{ is piecewise constant} \}
\]
on elements of the nonmortar triangulation \( T_L(\delta_{m(j)}) \). \( \) (3)

The dimension of \( S_L(\delta_{m(j)}) \) is equal to the number of midpoints on the \( \delta_{m(j)} \), i.e. to the number of elements on \( \delta_{m(j)} \). For each nonmortar edge \( \delta_{m(j)} \), let \( Q_{L,\delta_{m(j)}} : L^2(\gamma_m) \rightarrow S_L(\delta_{m(j)}) \) be the \( L^2 \)-projection operator defined by

\[
(Q_{L,\delta_{m(j)}} v, w)_{L^2(\delta_{m(j)})} = (v, w)_{L^2(\delta_{m(j)})}, \forall w \in S_L(\delta_{m(j)}),
\]

where \( (\cdot, \cdot)_{L^2(\delta_{m(j)})} \) denotes the \( L^2 \) inner product in the space \( L^2(\delta_{m(j)}) \).

We now define the following mortar finite element space for the nonconforming P1 finite element, associated with the finest level \( L \):

\[
V_L = \{ v_L | v_L \in \tilde{V}_L, Q_{L,\delta_{m(j)}}(v_L|_{\delta_{m(j)}}) = Q_{L,\delta_{m(j)}}(v_L|_{\gamma_m^{(i)}}), \text{ for } \forall \gamma_m \in \Gamma \}.
\]

Let

\[
a_{L,i}(u,v) := \sum_{K \in T_{L,i}} \int_K \nabla u \cdot \nabla v dx \ \forall u,v \in V_{L,i},
\]

and

\[
a_L(u,v) := \sum_{i=1}^{N} a_{L,i}(u,v).
\]

Then the discrete problem of (2) is to find \( u_L \in \tilde{V}_L \) such that

\[
a_L(u_L,v_L) = (f,v_L) \ \forall v_L \in V_L,
\]

(5)
where \((f, v_L) = \sum_{i=1}^{N} \int_{\Omega_i} f v_L \, dx\).

We define
\[
\|v\|_{L,i}^2 := a_{L,i}(v, v) \quad \text{and} \quad \|v\|_{L}^2 := \sum_{i=1}^{N} \|v\|_{L,i}^2, \quad \forall v \in V_L.
\]

From [17], we know that the discrete problem (5) has a unique solution. Moreover, the following error estimate can be found in [17]: Let \(u\) and \(u_L\) be the solutions of (2) and (5), respectively, then
\[
\|u - u_L\|_{L}^2 \leq Ch_L^2 \sum_{i=1}^{N} |u|_{H^2(\Omega_i)}^2.
\]

Next we define the operator \(A_L : V_L \rightarrow V_L\) as follows:
\[
(A_L v_L, w_L) = a_L(v_L, w_L) \quad \forall v_L, w_L \in V_L.
\]

Then (5) can be rewritten as:
\[
A_L u_L = f_L, \quad (6)
\]

where \(f_L = Q_L f\), and \(Q_L\) is the \(L^2\)-projection from space \(L^2(\Omega)\) to \(V_L\). In the following two sections, we try to design our multilevel preconditioner for (6).

### 3 A multilevel preconditioner for the conforming P1 mortar FE

In this section, we briefly describe the mortar method for the conforming P1 finite element, cf. [6], and then formulate the multilevel preconditioner for the method, developed in [13]. In the following section, we will use this preconditioner to construct our multilevel preconditioner for the discrete system (6).

Let \(\tilde{W}_{l,i}\) be the piecewise linear continuous finite element space over the triangulation \(T_{l,i}\), whose functions have zero trace on \(\partial \Omega\). Let
\[
\tilde{W}_l = \prod_{i=1}^{N} \tilde{W}_{l,i},
\]
for all \(l = 1, \ldots, L\). Obviously, the subspaces \(\{\tilde{W}_l\}\) are nested, that is, we have
\[
\tilde{W}_1 \subseteq \cdots \subseteq \tilde{W}_L.
\]

We now describe the conforming P1 mortar finite element method on the finest level \(L\). The basic idea of this mortar element method is to replace the point-wise continuity across the interfaces by a weaker one. To specify this interface condition, we should introduce
some trace spaces. Let \( \bar{M}_L(\gamma_{m(i)}) \) and \( \bar{M}_L(\delta_{m(j)}) \) be the piecewise continuous linear function space corresponding to the triangulation \( T_L(\gamma_{m(i)}) \) and \( T_L(\delta_{m(j)}) \), respectively. In addition, we define an auxiliary test space \( M_L(\delta_{m(j)}) \) as a subspace of the nonmortar space \( \bar{M}_L(\delta_{m(j)}) \) such that its functions are constants on elements that intersect the ends of \( \delta_{m(j)} \). Based on the above preparation, we can now define the following conforming P1 mortar finite element space

\[
W_L = \{ v_L \in \bar{W}_L\ | \forall \delta_{m(j)} \subset \Gamma, \int_{\delta_{m(j)}} (v_{L,i} - v_{L,j}) \varphi ds = 0, \forall \varphi \in M_L(\delta_{m(j)}) \}.
\]

The mortar finite element approximation of the problem (2) on the finest level \( L \), is to find \( u_L \in W_L \) such that

\[
\hat{a}_L(u_L, v_L) = (f, v_L), \quad \forall v_L \in W_L. \tag{7}
\]

where

\[
\hat{a}_L(u_L, v_L) := \sum_{i=1}^{N} \int_{\Omega_i} \nabla u_L \cdot \nabla v_L dx.
\]

We define

\[
|v|_L := \hat{a}_L(v, v), \quad \forall v \in W_L.
\]

It is shown in [6] that (7) has a unique solution.

Let \( \hat{Q}_{L, \delta_{m(j)}} : L^2(\gamma_m) \to M_L(\delta_{m(j)}) \) be the \( L^2 \)-projection, i.e.,

\[
(\hat{Q}_{L, \delta_{m(j)}} v, w_L) = (v, w_L) \quad \forall w_L \in M_L(\delta_{m(j)}).
\]

It is known that, cf. [8],

\[
\| (I - \hat{Q}_{L, \delta_{m(j)}}) v \|_{L^2(\gamma_m)} \leq Ch^\frac{1}{2} |v|_{H^\frac{1}{2}(\gamma_m)} \quad \forall v \in H^\frac{1}{2}(\gamma_m). \tag{8}
\]

Define the operator \( \hat{A}_L : W_L \to W_L \) as follows:

\[
(\hat{A}_L v_L, w_L) = \hat{a}_L(v_L, w_L) \quad \forall v_L, w_L \in W_L.
\]

In [13], an effective multilevel preconditioner \( \hat{B}_L \) for \( \hat{A}_L \) has been designed. In the following, we briefly describe this preconditioner.

First, we define the space \( \hat{S}_l(\delta_{m(j)}) \) by

\[
\hat{S}_l(\delta_{m(j)}) = \{ v \mid v \text{ is a piecewise linear continuous function on } T(\delta_{m(j)}),
\text{ and vanishing at the endpoints of } \delta_{m(j)} \}.\]
Accordingly, we define a projection operator \( \pi_{L,\delta_{m(j)}} : L^2(\Omega) \rightarrow \hat{S}_L(\delta_{m(j)}) \) as follows [6, 14]:

\[
\int_{\delta_{m(j)}} (\pi_{L,\delta_{m(j)}} v) \chi ds = \int_{\delta_{m(j)}} u \chi ds, \quad \forall \chi \in \hat{S}_L(\delta_{m(j)}).
\]

This projection is known to be stable in \( L^2(\gamma_m) \) and \( H^{1/2}_0(\gamma_m) \) [6, 8], that is

\[
\| \pi_{L,\delta_{m(j)}} v \|_{L^2(\delta_{m(j)})} \leq C \| v \|_{L^2(\gamma_m)},
\]

\[
\| \pi_{L,\delta_{m(j)}} v \|_{H^{1/2}_0(\delta_{m(j)})} \leq C \| v \|_{H^{1/2}_0(\gamma_m)}.
\]

As in [13], we introduce an extension operator \( Z_{\gamma_m} : L^2(\gamma_m) \rightarrow \tilde{W}_{L,i} \) on each interface \( \gamma_m \) as follows:

\[
Z_{\gamma_m} v := \sum_{l=1}^{L} F_{l,\delta_{m(j)}} (P_{l,\delta_{m(j)}} - P_{l-1,\delta_{m(j)}}) \pi_{L,\delta_{m(j)}} v, \quad \forall v \in L^2(\gamma_m),
\]

where \( F_{l,\delta_{m(j)}} \) is the trivial zero extension operator on the level \( l \) with respect to the non-mortar subdomain, and \( P_{l,\delta_{m(j)}} \) is the \( L^2 \) projection operator from \( \hat{S}_L(\delta_{m(j)}) \) to \( \hat{S}_L(\delta_{m(j)}) \), where we set \( P_{0,\delta_{m(j)}} := 0 \).

Next we define an intergrid transfer operator \( Z_i : \tilde{W}_{l,i} \rightarrow W_L \) in terms of \( Z_{\gamma_m} \) by means of

\[
Z_i v := \begin{cases} 
  v - \sum_{\gamma_{m(j)} \subseteq \partial T_i} Z_{\gamma_{m(j)}} v|_{\Omega_i \cap \gamma_{m(j)}}, & \text{on } \Omega_i, \\
  Z_{\gamma_{m(i)}} v|_{\Omega_i \cap \gamma_{m(i)}}, & \text{on } \Omega_i \cap \gamma_{m(i)}, \\
  0, & \text{elsewhere},
\end{cases}
\]

where \( \Omega_i \cap \gamma_{m(i)} \) denotes the non-mortar subdomains with \( \gamma_{m(i)} \subseteq \partial \Omega_i \) as mortar edges. It is easy to check that \( Z_i v \in W_L \). Based on the intergrid transfer operator, we can provide a decomposition of the space \( W_L \), that is

\[
W_L = \sum_{l=1}^{L} \sum_{i=1}^{N} Z_i \tilde{W}_{l,i}.
\]

We now introduce an inexact bilinear form \( b_{l,i}(\cdot, \cdot) : \tilde{W}_{l,i} \times \tilde{W}_{l,i} \rightarrow R \) as

\[
b_{l,i}(v_{l,i}, w_{l,i}) := \sum_{x \in N_i} v_{l,i}(x) w_{l,i}(x), \quad v_{l,i}, w_{l,i} \in \tilde{W}_{l,i},
\]

where \( N_i \) is the set of conforming P1 nodal points of the triangulation \( T_{i,j} \) in \( \tilde{\Omega}_i \). The corresponding projection operator \( \bar{T}_{l,i} : W_L \rightarrow \tilde{W}_{l,i} \) can be defined by

\[
b_{l,i}(\bar{T}_{l,i} v, v_{l,i}) := a_L(v, Z_i v_{l,i}), \quad v_{l,i} \in \tilde{W}_{l,i}.
\]

Let

\[
T_{l,i} := Z_i \bar{T}_{l,i} : W_L \rightarrow W_L.
\]
Then the preconditioned system can be expressed by

\[ T := \hat{B}_L \hat{A}_L := \sum_{l=1}^{L} \sum_{i=1}^{N} T_{l,i}. \]

The multilevel preconditioner \( \hat{B}_L \) has the following algebraic form (cf. [13] for details)

\[ \hat{B}_L := \sum_{l=1}^{L} \sum_{i=1}^{N} Z_i R_{l,i} (Z_i R_{l,i})^T, \]

where \( Z_i \) is the algebraic representation of \( Z_i \), and \( R_{l,i} \) is the prolongation matrix from \( \tilde{W}_{l,i} \rightarrow \tilde{W}_{L,i} \). Adding a coarse space solver, based on a continuous vertex basis function for each subdomain vertex, we have the following theorem on the preconditioned system, cf. [13], where it states that the condition number is proportional to the square of the number of refinement levels.

**Theorem 1.** There holds that [13]

\[ c \hat{a}_L(v, v) \leq \hat{a}_L(\hat{B}_L \hat{A}_L v, v) \leq C L^2 \hat{a}_L(v, v), \forall v \in W_L. \]

**Remark 1.** For the non crosspoint case, we have [13]

\[ c \hat{a}_L(v, v) \leq \hat{a}_L(\hat{B}_L \hat{A}_L v, v) \leq C L \hat{a}_L(v, v), \forall v \in W_L. \]

### 4 A multilevel preconditioner for the nonconforming P1 mortar FE

In this section, we will use the idea in [26, 19] to construct our multilevel preconditioner for the mortar element method in Section 2. At first, we introduce a transfer operator from the space \( W_L \) to \( V_L \).

Define an operator \( \Xi_{L, m(i)} : \tilde{V}_L \rightarrow \tilde{V}_L \) by

\[ (\Xi_{L, m(i)}(v))(x) = \begin{cases} (Q_L, m(i))(v|_{\gamma_{m(i)}} - v|_{\delta_{m(i)}})(x) & x \in \delta_{L, m(i)}^C, \\ 0 & \text{all other CR nodes}. \end{cases} \]  

Then for any \( v \in \tilde{W}_L \subset \tilde{V}_L \), let

\[ v^* = v + \sum_{m=1}^{M} \Xi_{L, m(i)}(v). \]

It is easy to check that \( v^* \in V_L \). Based on this observation, we define the following transfer operator \( I_L : W_L \rightarrow V_L \), which will appear in the following multilevel algorithm:

\[ I_L v = v + \sum_{m=1}^{M} \Xi_{L, m(i)}(v), \forall v \in W_L. \]  

For the operator \( I_L \), we have
Lemma 1. For any $v \in W_L$, it holds that

$$
\|v - I_L v\|_{L^2(\Omega)} \leq Ch_L |v|_L, \quad (12)
$$

$$
\|I_L v\|_L \leq C|v|_L, \quad (13)
$$

Proof. At first, we prove (12). It is easy to see that

$$
\|v - I_L v\|_{L^2(\Omega)} \leq C \sum_{m=1}^{M} \|\Xi_{L,\delta m(j)}(v)\|_{L^2(\Omega)}^2. \quad (14)
$$

For each nonmortar edge $\delta_{m(j)}$, we can derive

$$
\|\Xi_{L,\delta m(j)}(v)\|_{L^2(\Omega)}^2 \leq Ch_L^2 \sum_{x \in \delta_{L,R,m(j)}} (\Xi_{L,\delta m(j)}(v)(x))^2
$$

$$
= Ch_L^2 \sum_{x \in \delta_{L,R,m(j)}} (Q_{L,\delta m(j)}(v|_{\gamma_{m(i)}} - v|_{\delta m(j)}))^2(x)
$$

$$
\leq Ch_L \|Q_{L,\delta m(j)}(v|_{\gamma_{m(i)}} - v|_{\delta m(j)})\|_{L^2(\gamma_m)}^2
$$

$$
\leq Ch_L \|v|_{\gamma_{m(i)}} - v|_{\delta m(j)}\|_{L^2(\gamma_m)}^2. \quad (15)
$$

On the other hand, owing to $v \in W_L$ and (8), we have

$$
\|v|_{\gamma_{m(i)}} - v|_{\delta m(j)}\|_{L^2(\gamma_m)}^2
$$

$$
\leq 2(\|v|_{\gamma_{m(i)}} - \hat{Q}_{L,\delta m(j)}(v|_{\gamma_{m(i)}}))^2_{L^2(\gamma_m)}
$$

$$
+ \|v|_{\delta m(j)} - \hat{Q}_{L,\delta m(j)}(v|_{\delta m(j)}))^2_{L^2(\gamma_m)}
$$

$$
\leq Ch_L \|v|_{\gamma_{m(i)}}^2_{H^1(\gamma_m)} + \|v|_{\delta m(j)}^2_{H^1(\gamma_m)}
$$

$$
\leq Ch_L \|v\|_{H^1(\Omega_i)}^2 + \|v\|_{H^1(\Omega_j)}^2. \quad (16)
$$

Combining the above three inequalities, (14)-(16), gives (12). We now prove (13). Using a similar argument as in the proof of (12), we can derive

$$
\|v - I_L v\|_L \leq C|v|_L,
$$

which, together with the triangle inequality, yields (13). \qed

Next we describe a basis for the space $V_L$. Let $\{\tilde{\phi}^k_L \mid k = 1, ..., n_L\}$ be the nodal basis of $V_L$. By the operator $\Xi_{L,\delta m(j)}$, the basis of $V_L$ consists of functions of the form

$$
\phi^k_L = \tilde{\phi}^k_L + \sum_{m=1}^{M} \Xi_{L,\delta m(j)}(\tilde{\phi}^k_L). \quad (17)
$$

From above definition, we can see that there exist two kinds of basis function of space $V_L$:
• Case 1. \( \phi_L^k \) and \( \tilde{\phi}_L^k \) corresponding to all nodal points in the interior of each subdomain, except those belonging to the triangles having at least one edge on a mortar, are the same.

• Case 2. \( \phi_L^k \) corresponding to all nodal points of each mortar edge \( \gamma_m \subset \Gamma \), and those nodal points in the interior of each subdomain, belonging to the triangles having at least one edge on a mortar, are defined by (17).

It is easy to check that \( \phi_L^k \) at all nodes which are in the interior of each nonmortar edge \( \delta_{m(i)} \subset \Gamma \) are equal to zero. One can see that these \( \phi_L^k \) from Case 1 and Case 2 form a basis of \( V_L \). Moreover, by the definition of mortar space and (17), we know that the basis function \( \phi_L^k \) has local support as the usual standard finite element nodal basis. This is not the case for the conforming \( P_1 \) mortar finite element [14].

We now define a smoothing operator \( R_L \), which corresponds to the Richardson iteration, as the following.

\[
R_L v = \sum_{k=1}^{n_L} (v, \phi_L^k) \phi_L^k \quad \forall v \in V_L. \tag{18}
\]

**Lemma 2.** It holds that

\[
ch_L^{-2} (v, v) \leq (R_L v, v) \leq Ch_L^{-2} (v, v) \quad \forall v \in V_L.
\]

**Proof.** We only need to prove

\[
\| \phi_L^k \|_{L^2(\Omega)} = O(h_L^2). \tag{19}
\]

Indeed, for the basis function \( \phi_L^k \) which are in the interior of each subdomain, using a standard scaling argument, it is easy to check that \( \| \phi_L^k \|_{L^2(\Omega)} = O(h_L^2) \). For the second case basis function, by the definition (4.4), we know

\[
\| \phi_L^k \|_{L^2(\Omega)} \leq \| \tilde{\phi}_L^k \|_{L^2(\Omega)} + \| \Xi L_{\delta_{m(i)}} (\tilde{\phi}_L^k) \|_{L^2(\Omega)} \\
\leq Ch_L^2 + Ch_L \| \tilde{\phi}_L^k \|_{L^2(\delta_{m(i)})} \\
\leq Ch_L^2 + Ch_L \| \tilde{\phi}_L^k \|_{L^2(\gamma_m(i))} \\
\leq Ch_L^2.
\]

Then for all basis function in \( V_L \), (19) is true. Using a fully same argument as in [27, 28], we know that Lemma 2 is valid.
Finally we can define our multilevel preconditioner for the mortar-type CR element method as follows:

\[ B_L = R_L + I_L B_L I_L' , \]  
\[ (I_L' v, w) = (v, I_L w) \quad \forall v \in V_L, w \in W_L. \]  

5 Condition number estimate

By the lemmas 4.1 and 4.2 and the abstract theorem developed in [26, 15], we know that the following theorem is true.

**Theorem 2.** Let \( B_L \) be defined as in (20), and assume that there exists a linear operator \( J_L : V_L \rightarrow W_L \) such that

\[ |J_L v|_L \leq C \|v\|_L, \quad \forall v \in V_L, \]  
and

\[ \|v - I_L J_L v\|_{L^2(\Omega)} \leq C h_L \|v\|_L \quad \forall v \in V_L. \]  

Then, we have

\[ c a_L(v, v) \leq a_L(B_L A_L v, v) \leq C L^2 a_L(v, v) \quad \forall v \in V_L. \]  

For the non crosspoint case, we have

\[ c a_L(v, v) \leq a_L(B_L A_L v, v) \leq C L a_L(v, v) \quad \forall v \in V_L. \]  

**Proof.** The proof follows immediately from the abstract framework of the auxiliary space method [26] and Theorem 1. \( \square \)

It follows from Theorem 2 that, if we can show (21) and (22) for the spaces \( V_L \) and \( W_L \), then \( B_L \) will be a good multilevel preconditioner for the CR mortar finite element.

At first, we introduce a transfer operator \( E_L \) from \( V_L \) to \( W_L \), which is similar to the one constructed in [29]. On each subdomain, define an operator \( E_{L,i} : V_{L,i} \rightarrow W_{L,i} \) as follows:

- **Case 1.** If \( x \in \Omega_{L,i}^P \), and \( x \notin \partial \Omega \), then

\[ (E_{L,i} v)(x) = \frac{1}{q(x)} \sum_{K_i} v|_{K_i}(x) \]
where $\Omega_{L,i}^P$ is the set of the vertices of the triangulation $T_{L,i}$ that are in $\bar{\Omega}_i$, and the sum is taken over all triangles $K \in T_{L,i}$ having $x$ as their common vertex, and $q(x)$ being the number of those triangles.

- Case 2. If $x \in \partial \Omega \cap \partial \Omega_{L,i}^P$, then

$$\mathcal{E}_{L,i}(v)(x) = 0,$$

where $\partial \Omega_{L,i}^P$ is the set of vertices of the triangulation $T_{L,i}$ that are on $\partial \Omega_i$.

For the operator $\mathcal{E}_{L,i}$, we have

**Lemma 3.** For any $v \in V_{L,i}$, it holds that

$$|\mathcal{E}_{L,i}v|_{H^1(\Omega_i)} \leq C\|v\|_{L,i},$$  \hspace{1cm} (23)

$$\|\mathcal{E}_{L,i}v - v\|_{L^2(\Omega_i)} \leq Ch_L\|v\|_{L,i},$$  \hspace{1cm} (24)

$$\|\mathcal{E}_{L,i}v - v\|_{L^2(\gamma_m)} \leq Ch_L^{1/2}\|v\|_{L,i},$$  \hspace{1cm} (25)

where $\gamma_m$ is an edge of $\Omega_i$.

**Proof.** The proof of (23) and (24) can be found in [21, 22, 30]. For the proof of (25), we refer to Lemma 3.3 in [17]. \hfill \square

Based on the operator $\mathcal{E}_{L,i}$, we define an intergrid transfer operator $\mathcal{E}_L : \tilde{V}_L \rightarrow \tilde{W}_L$ as follows: for any $v = (v_1, \ldots, v_N) \in \tilde{V}_L$,

$$\mathcal{E}_Lv = (\mathcal{E}_{L,1}v_1, \ldots, \mathcal{E}_{L,N}v_N) \in \tilde{W}_L.$$

We then define the transfer operator $J_L$ as follows:

$$J_Lv = \mathcal{E}_Lv + \sum_{m=1}^M \tilde{\mathcal{E}}_{L,\delta_{m(j)}}(\mathcal{E}_Lv),$$ \hspace{1cm} (26)

where $\tilde{\mathcal{E}}_{L,\delta_{m(j)}} : \tilde{W}_L \rightarrow \tilde{W}_L$ is given by

$$\tilde{\mathcal{E}}_{L,\delta_{m(j)}}(v)(x) = \begin{cases} (\pi_{L,\delta_{m(j)}}(v|_{\gamma_m(i)} - v|_{\delta_{m(j)}}))(x) & x \in \delta_{L,m(j)}^P, \\ 0 & \text{all other CR nodes,} \end{cases}$$ \hspace{1cm} (27)

here $\delta_{L,m(j)}^P$ denotes the set of the vertices belonging to $\delta_{m(j)}$. It is easy to check that $J_Lv \in W_L$. 

12
For any \( v \in V_L \), it holds that
\[
\|v - J_L v\|_{L^2(\Omega)} \leq C h_L \|v\|_L, \quad (28)
\]
\[
|J_L v|_L \leq C \|v\|_L. \quad (29)
\]

**Proof.** We prove (28) first. Using Lemma 3, we get
\[
\|v - J_L v\|_{L^2(\Omega)} \leq C(\|v - E_L v\|_{L^2(\Omega)}^2 + \sum_{m=1}^{M} \|\tilde{E}_L,\delta_{m(i)}(E_L v)\|_{L^2(\Omega)}^2)
\]
\[
\leq C(h_L^2 \|v\|_2^2 + \sum_{m=1}^{M} \|\tilde{E}_L,\delta_{m(i)}(E_L v)\|_{L^2(\Omega)}^2). \quad (30)
\]

For each nonmortar edge \( \delta_{m(i)} \),
\[
\|\tilde{E}_L,\delta_{m(i)}(E_L v)\|_{L^2(\Omega)}^2 \leq Ch_L^2 \sum_{x \in E_{\delta_{m(i)}}} (\tilde{E}_L,\delta_{m(i)}(E_L v)(x))^2
\]
\[
= Ch_L^2 \sum_{x \in E_{\delta_{m(i)}}} (\tau_{L,\delta_{m(i)}}((E_L v)_{\gamma_{m(i)}} - (E_L v)_{\delta_{m(i)}}))^2(x)
\]
\[
\leq Ch_L \|\tau_{L,\delta_{m(i)}}((E_L v)_{\gamma_{m(i)}} - (E_L v)_{\delta_{m(i)}})\|_{L^2(\gamma_{m(i)})}^2
\]
\[
\leq Ch_L \|((E_L v)_{\gamma_{m(i)}} - (E_L v)_{\delta_{m(i)}})\|_{L^2(\gamma_{m(i)})}^2
\]
\[
\leq Ch_L (\|v|_{\gamma_{m(i)}} - v|_{\delta_{m(i)}}\|_{L^2(\gamma_{m(i)})}^2 + \|v|_{\delta_{m(i)}} - (E_L v)_{\delta_{m(i)}}\|_{L^2(\delta_{m(i)})}^2)
\]
\[
:= Ch_L (K_1 + K_2). \quad (31)
\]

It follows immediately from Lemma 3 that
\[
K_2 \leq Ch_L \|v\|_{L^2(\gamma_{m(i)})}^2. \quad (32)
\]

In the following, we estimate the term \( K_1 \). Since \( v \in V_L \), we get
\[
\|((E_L v)_{\gamma_{m(i)}} - v|_{\delta_{m(i)}}\|_{L^2(\gamma_{m(i)})}^2 \leq 2\|(E_L v)_{\gamma_{m(i)}} - Q_{L,\delta_{m(i)}}(v|_{\gamma_{m(i)}})\|_{L^2(\gamma_{m(i)})}^2
\]
\[
+ 2\|Q_{L,\delta_{m(i)}}(v|_{\delta_{m(i)}}) - v|_{\delta_{m(i)}}\|_{L^2(\delta_{m(i)})}^2. \quad (33)
\]

For the second term in the above inequality, we first note that
\[
\|Q_{L,\delta_{m(i)}}(v|_{\delta_{m(i)}}) - v|_{\delta_{m(i)}}\|_{L^2(\delta_{m(i)})}^2 = \sum_{e \in T_E(\delta_{m(i)})} \int_e (v - Q_E v)^2 ds, \quad (34)
\]
where $Q_e$ is the $L^2$ orthogonal projection onto one-dimensional space which consists of constant functions on an element $e$. Using a scaling argument, for any constant $c$, we have

$$\int_e (v - Q_ev)^2 \, ds \leq \int_e (v - c)^2 \, ds \leq ChL\|\bar{v} - c\|_{H^1(K)}^2 \leq ChL\|\bar{v}\|_{H^1(K)}^2 \leq ChL\|v\|_{H^1(K)}^2.$$

Now combining (34) and (35) we arrive at

$$\|Q_{L,\delta_m(j)}(v|_{\delta_m(j)}) - v|_{\delta_m(j)}\|_{L^2(\delta_m(j))} \leq C_{h_L}^{1/2}\|v\|_{L,\Omega}.$$  (36)

For the first term on the right side of (5.7), we deduce that

$$\|((E_{L,i}v)|_{\gamma_{m(i)}} - Q_{L,\delta_{m(i)}}(v|_{\gamma_{m(i)}})\|_{L^2(\gamma_{m(i)})}^2 \leq 2\|((E_{L,i}v)|_{\gamma_{m(i)}} - Q_{L,\delta_{m(i)}}((E_{L,i}v)|_{\gamma_{m(i)}})\|_{L^2(\gamma_{m(i)})}^2 + 2\|Q_{L,\delta_{m(i)}}((E_{L,i}v)|_{\gamma_{m(i)}} - v|_{\gamma_{m(i)}})\|_{L^2(\delta_{m(i)})}^2 \leq F_1 + F_2.$$  (37)

Applying a similar argument as in the proof of (36), and Lemma 3 we get

$$F_1 \leq ChL\|E_{L,i}v\|_{H^1(\Omega_i)}^2 \leq ChL\|v\|_{L,i}^2.$$  (38)

For $F_2$, using Lemma 3, we have

$$F_2 \leq C\|((E_{L,i}v)|_{\gamma_{m(i)}} - v|_{\gamma_{m(i)}}\|_{L^2(\gamma_{m(i)})}^2 \leq ChL\|v\|_{L,i}^2,$$  (39)

which, together with (30)-(33), and (36)-(38), gives (28).

We now prove (29). In fact, by the definition of $E_{L,i}$ and Lemma 3, we deduce

$$\|v - J_Lv\|_L^2 \leq \|v - E_{L,i}v\|_L^2 + \sum_{m=1}^M \|\tilde{E}_{L,\delta_{m(j)}}(E_{L,i}v)\|_L^2 \leq C(\|v\|_L^2 + \sum_{m=1}^M \|\tilde{E}_{L,\delta_{m(j)}}(E_{L,i}v)\|_L^2).$$  (40)

Using a similar argument as in the proof of (1), we can derive

$$\|\tilde{E}_{L,\delta_{m(j)}}(E_{L,i}v)\|_L^2 \leq C\|v\|_{L,\Omega_i \cup \Omega_j}^2.$$  (41)

Combining (40) with (41) yields (29).
Based on Theorem 3, we know that (21) is true. On the other hand, by Lemma 1 and Theorem 3, we have

\[
\|v - I_L J_L v\|_{L^2(\Omega)} \leq \|v - J_L v\|_{L^2(\Omega)} + \|(I - I_L) J_L v\|_{L^2(\Omega)} \\
\leq Ch_L \|v\|_L + Ch_L |J_L v|_L \\
\leq Ch_L \|v\|_L,
\]

which is (5.2). Theorem 2 is thereby completed.

Remark 2. Using the fully same technique developed in this paper, we can construct an effective multilevel preconditioner for the mortar Wilson element method, which was proposed in [23].

6 Numerical results

We present in this section the numerical results from our experiments. The model problem is defined on an unit square domain, \(\Omega = (0, 1)^2\), with the forcing function \(f\) equals to \(2\pi^2 \sin(\pi x) \sin(\pi y)\), and a homogeneous Dirichlet boundary condition resulting in the exact solution \(u\) equals to \(\sin(\pi x) \sin(\pi y)\). At first, the domain \(\Omega\) is partitioned into a \(d_x \times d_y\) rectangular subdomains (subregions). Initially, each subdomain \(\Omega_i\) is triangulated using Matlab’s discretization routine \texttt{initmesh} with the mesh size parameter \(h_{\text{max}}\) as an input parameter. The parameter \(h_{\text{max}}\) is one of two real numbers defining two different mesh sizes distributed among the subdomains in a checkerboard fashion. Starting from the initial triangulation \((l = 1)\), we decompose each triangle into four subtriangles in each refinement step. The resulting grid is nonmatching across all interfaces, and we use the CR mortar finite element [17] for the discretization of the model problem. The resulting discrete problem is solved using the Preconditioned Conjugate Gradients (PCG) method with the multilevel preconditioner for the CR mortar finite element proposed in this paper. For the smoothing operator we consider only the Richardson type here as it is simple, and turned out to be very effective for our case.

For our experiments, we consider two different partitions of the domain, one with two subdomains not resulting in any internal crosspoint, cf. Figure 1 (left picture), and one with nine subdomains resulting in four internal crosspoints, cf. Figure 1 (right picture). The numerical results are presented in tables 1-2, showing, for each test case, the following quantities: number of levels (‘levels’), number of interior degrees of freedom (‘dofs’), condition
Figure 1: Initial nonmatching grids as a result of calling Matlab’s triangulation routine \texttt{initmesh} separately for each subdomain, with the input parameter $h_{\text{max}} = 0.2$ and $0.35$ (on the left) and $h_{\text{max}}$ alternatingly equal to $0.15$ and $0.30$ (on the right).

number estimate ('$\kappa$') with the number of iterations ('iter', inside parentheses) required to reduce the residual norm by the factor $10^{-6}$, and the $L^2$ norm of the error ('error$_{L^2}$') in the computed solution. For the comparison, we also include the corresponding numerical results of applying to our model problem, the original multilevel preconditioner for the P1 mortar finite element as proposed in [13].

Table 1: The multilevel Schwarz for the P1 mortar element and the proposed multilevel Schwarz for the CR mortar element on $d_x \times d_y = 2 \times 1$ nonmatching grids.

<table>
<thead>
<tr>
<th># levels</th>
<th># dofs</th>
<th>error$_{L^2}$</th>
<th>$\kappa$ (# iter)</th>
<th># dofs</th>
<th>error$_{L^2}$</th>
<th>$\kappa$ (# iter)</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>377</td>
<td>4.269e-3</td>
<td>15.00 (21)</td>
<td>1208</td>
<td>1.131e-3</td>
<td>19.21 (26)</td>
</tr>
<tr>
<td>4</td>
<td>1585</td>
<td>1.069e-3</td>
<td>17.58 (24)</td>
<td>4912</td>
<td>2.839e-4</td>
<td>20.51 (30)</td>
</tr>
<tr>
<td>5</td>
<td>6497</td>
<td>2.672e-4</td>
<td>19.46 (27)</td>
<td>19808</td>
<td>7.103e-5</td>
<td>21.72 (32)</td>
</tr>
<tr>
<td>6</td>
<td>26305</td>
<td>6.679e-5</td>
<td>20.95 (30)</td>
<td>79552</td>
<td>1.776e-5</td>
<td>22.90 (34)</td>
</tr>
</tbody>
</table>

Generally speaking, the condition number estimates and the iteration counts as seen from the tables, support our theory presented in this paper. The numerical results simply reflects the fact that the convergence behavior of the multilevel preconditioner for the CR mortar finite element is similar to that of the original multilevel preconditioner for the conforming P1 mortar finite element. Table 2 corresponds to the initial discretization shown in the left picture of Figure 1, representing the case without any crosspoint. As seen from the table, the condition number estimates depend very mildly on the number of refinement levels. Table 1 corresponds to the initial discretization shown in the right picture of Figure 1, where there are four internal crosspoints. We observe asymptotically a quadratic and a linear dependence of the condition number estimates and the iteration counts, respectively,
on the number of grid refinement levels, which is in agreement with our theory.

Table 2: The multilevel Schwarz for the P1 mortar element and the proposed multilevel
Schwarz for the CR mortar element on $d_x \times d_y = 3 \times 3$ nonmatching grids.

<table>
<thead>
<tr>
<th># levels</th>
<th>Multilevel for P1 Mortar</th>
<th>Multilevel for CR Mortar</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td># levels</td>
<td># dofs</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>597</td>
</tr>
<tr>
<td>4</td>
<td>4</td>
<td>2525</td>
</tr>
<tr>
<td>5</td>
<td>5</td>
<td>10413</td>
</tr>
<tr>
<td>6</td>
<td>6</td>
<td>42317</td>
</tr>
</tbody>
</table>

We make some additional remarks here. Evidently, the number of degrees of freedom
on a fixed level seems three times as many for the CR mortar finite element as it is for the
P1 mortar finite element, in other words, the problem in the CR mortar case is three times
as large as in the conforming P1 case. Since, in both cases, we know that the error in the
$L^2$ norm varies as the $h^2$ ($h$ is the mesh size), it is tempting to expect that the error in the
CR mortar case is three times as small as that in the conforming P1 mortar case. In fact,
the numerical results presented in tables 1-2, do exhibit this fact.

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